Algebra objects in the representation category of Taft algebras

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Outline

1. Introduction
2. Module Categories
3. Tensor Algebras in Tensor Categories
4. The Taft Algebra Case
5. Future Directions

Introduction

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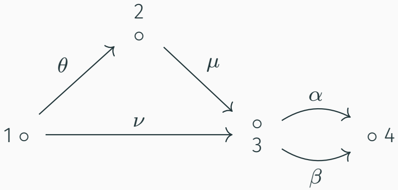
Quiver

A quiver Q is a directed graph, formally described by:

1. a collection of vertices Q_0,
2. a collection of edges (or arrows) Q_1,
3. maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ describing the source and target of an arrow.

Quiver

A quiver is a directed graph with a collection of vertices, a collection of arrows, and maps describing the initial and terminal vertices of an arrow.



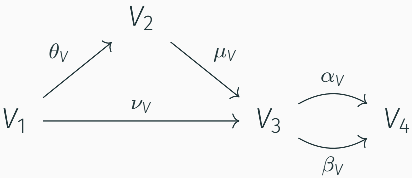
Quiver Representations

Representations of a Quiver

A representation V of a quiver Q is the following collection of data:

1. for every vertex $x \in Q_0$, a vector space V_x,
2. for every arrow $x \xrightarrow{\alpha} y$ a linear map $\alpha_V : V_x \to V_y$.

Example:



Morphisms of Representations

A morphism of representations $f : V \to W$ is a collection of linear mas $f_x : V_x \to W_x$ such that the following diagram commutes

$$\begin{array}{ccc} V_x & \xrightarrow{\alpha_V} & V_y \\ {\scriptstyle f_x}\downarrow & & \downarrow{\scriptstyle f_y} \\ W_x & \xrightarrow{\alpha_W} & W_y \end{array}$$

for all arrows $x \xrightarrow{\alpha} y$ in the quiver Q.

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Path Algebra

The path algebra $\Bbbk Q$ of a quiver Q is the algebra generated by paths in Q (including paths of length 0, that is, orthogonal idempotents e_x for each vertex $x \in Q_0$).

The algebra $\Bbbk Q$ is an associative algebra with unit $1 = \sum e_x$.

The vector space $\Bbbk Q_1$ generated by arrows in Q is a bimodule over the algebra $\Bbbk Q_0$ generated by the orthogonal idempotents.

The path algebra $\Bbbk Q$ can be viewed as the tensor algebra

$$\Bbbk Q_0 \oplus \Bbbk Q_1 \oplus \Bbbk Q_2 \oplus \dots$$

Theorem [Sch16]

The category of representations of a quiver Q is equivalent to the category $\Bbbk Q-\text{mod}$ of left modules over the path algebra $\Bbbk Q$.

Theorem

Let A be a finite dimensional associative algebra.

- A is Morita equivalent to some basic algebra A_b
- (Garbriel's theorem) There exists a quiver Q and an ideal $\mathcal{I} \subset \Bbbk Q$ such that $A_b \cong \Bbbk Q/\mathcal{I}$.

Setup: Let A and B be finite-dimensional $\Bbbk$-algebras.

Definition. A and B are Morita equivalent if there exist bimodules

$$ {}_BP_A \quad \text{and} \quad {}_AQ_B $$

such that

$$ P \otimes_A Q \;\cong\; B \quad \text{as } (B,B)\text{-bimodules,} $$

$$ Q \otimes_B P \;\cong\; A \quad \text{as } (A,A)\text{-bimodules.} $$

Induced Equivalences:

$$P \otimes_A - : \mathrm{Mod}\text{-}A \longrightarrow \mathrm{Mod}\text{-}B$$

$$Q \otimes_B - : \mathrm{Mod}\text{-}B \longrightarrow \mathrm{Mod}\text{-}A$$

are inverse equivalences of module categories.

Example: A and $M_n(A)$ are Morita equivalent for all $n \in \mathbb{Z}_+$.

Restatement of the above result:

Theorem

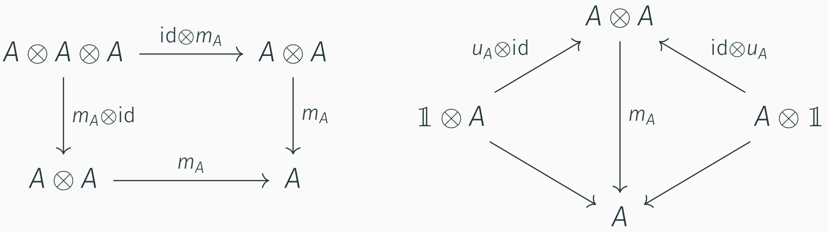
Let A be an algebra in $\text{vec}_{\mathbb{k}}$, the category of finite dimensional $\mathbb{k}-$vector spaces. There exists a quiver Q and an ideal $\mathcal{I} \subset \mathbb{k}Q$ such that the categories $A - \text{mod}$ and $\mathbb{k}Q/\mathcal{I} - \text{mod}$ are equivalent.

Are there any interesting generalizations of this result to arbitrary tensor categories?

Module Categories

Algebras in Tensor Categories

An algebra in a tensor category $\mathcal{C}$ is an object A in $\mathcal{C}$ along with morphisms $m_A : A \otimes A \to A$ and $u_A : \mathbb{1} \to A$ such that the following diagrams are commutative:



Examples

1. The unit object $\mathbb{1}$ in any tensor category $\mathcal{C}$ is an algebra.
2. An algebra in the category $\mathrm{Rep}(H)$ is an H-module algebra.

Constructing Left Module Categories from Algebras: For an algebra A in an abelian monoidal category $\mathcal{C}$, construct the category $\text{Mod}_{\mathcal{C}}(A)$ of right A-modules in $\mathcal{C}$ whose

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  $$a_M : M \otimes A \to M$$

  is an action morphism.

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- **Objects** are pairs (M, a_M) where M is an object in $\mathcal{C}$

  $$a_M : M \otimes A \to M$$

  is an action morphism.
- **Morphisms** are those morphisms of $\mathcal{C}$ which commute with the action morphisms.

$$\begin{array}{ccc} M_1 \otimes A & \xrightarrow{f \otimes \mathrm{id}} & M_2 \otimes A \\ \downarrow{\scriptstyle a_1} & & \downarrow{\scriptstyle a_2} \\ M_1 & \xrightarrow{\quad f \quad} & M_2 \end{array}$$

Tensor Algebras in Tensor Categories

Fix an exact algebra A in $\mathcal{C}$ and an indecomposable $A-$bimodule M in $\mathcal{C}$. The $\mathcal{C}-$tensor algebra $T_A(M)$ is given by

$$T_A(M) = A \oplus M \oplus (M \otimes_A M) \oplus \dots$$

with multiplication given by the natural maps

$$(M^{\otimes_A n}) \otimes_A (M^{\otimes_A m}) \to M^{\otimes_A (n+m)}$$

and the unit map is induced from the injection $A \hookrightarrow T_A(M)$.

Restrict to the case when the algebra A is a direct sum

$$A = A_1 \oplus A_2$$

of indecomposable algebras A_1 and A_2 in $\mathcal{C}$.

Studying $(A, A)-$bimodules amounts to studying $(A_i, A_j)-$bimodules for $i, j \in \{1, 2\}$. The case where the algebras A_i's are semisimple and commutative as $\Bbbk-$algebras is studied in Kinser-Oswald [KO21] and Etingof-Kinser-Walton [EKW19].

I am currently investigating $(A_i, A_j)-$bimodules, where the algebras are not necessarily commutative or semisimple as $\Bbbk-$algebras.

The Taft Algebra Case

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Taft Algebra H_l is generated by elements x and g subject to the relations

$$x^l = 0, \qquad g^l = 1, \qquad \text{and} \qquad xg = q^{-1}gx$$

and comultiplication

$$\Delta(g) = g \otimes g, \qquad \Delta(x) = x \otimes 1 + g \otimes x.$$

Theorem (Etingof and Ostrik) [EO03]

For any divisor d of l there is exactly one nonsemisimple indecomposable exact module category over Rep(H_l) with d simple objects and exactly one one-parameter family of semisimple indecomposable module categories over Rep(H_l) with d simple objects.

H_l-module algebra

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Let A be a simple H_l-module algebra, (i.e. contains no ideals invariant under the action of H_l) then the action of H_l on A allows us to introduce a filtration on A via

$$A_{-1} = 0 \quad \text{and} \quad A_i = \{a \in A |\, x \cdot a \in A_{i-1}\}.$$

This filtration has the following properties:

1. $x \cdot A_i \subset A_{i-1}$
2. $g \cdot A_i \subset A_i$
3. $A_i A_j \subset A_{i+j}$
4. $A_l = A$
5. for any $a \in A_i \setminus A_{i-1}$, $x \cdot a \in A_{i-1} \setminus A_{i-2}$
6. The algebra A_0 has no nontrivial $g-$invariant ideals.

The algebra A_0 is of the form $\Bbbk^{G/H}$ where H is a subgroup (unique, say, of order d) of $G = \langle g \rangle$. Let $e_s,\ s \in G/H$ denote the minimal central idempotents in A_0.

When $A \neq A_0$, A_1 contains a unique element y such that

$$x(y) = 1, \quad y = \sum_{s \in G/H} e_{gs} y e_s, \quad \text{and} \quad g(y) = q^{-1} y$$

The algebra A is generated by A_0 and y. The relations above give us

$$y^l = \lambda \cdot 1_{A_0}, \qquad \lambda \in \Bbbk^\times$$

Denote A by $A(d, \lambda)$.

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Nonsemisimple Indecomposable

In the case of $A = A_0 = \Bbbk^{G/H}$, the module category $\mathrm{Mod}_{\mathrm{Rep}(H_l)}(A)$ is nonsemisimple indecomposable with d isomorphism classes of simple objects.

Semisimple Indecomposable

When $A = A(d, \lambda)$, the module category $\mathrm{Mod}_{\mathrm{Rep}(H_l)}(A)$ is semisimple indecomposable with d isomorphism classes of simple objects.

When $A = A_0 = \Bbbk^{G/H}$:
Let $G = \langle g \rangle$ and H be its unique subgroup of order d. Then for $A = \Bbbk^{G/H}$ the action of H_l on A is given by:

1. $g \cdot (e_{g^j H}) = e_{g^{j+1} H}$ for all $g^j H \in G/H$
2. $x \cdot (e_{g^j H}) = 0$ for all $g^j H \in G/H$.

Compatibility of the two actions: Let M be a left H_l right A module. Then for all $m \in M$ and $a \in A$ we have

$$g(ma) = g(m)g(a)$$

$$x(ma) = x(m)a + g(m)x(a) = x(m)a$$

Nonsemisimple indecomposable $\mathrm{Rep}(H_l)-$module categories

Let $\mathcal{Q} = (Q, I)$ be a quiver with relations given by:

1. vertices, $Q_0 = \{i \mid i \in \mathbb{Z}/d\mathbb{Z}\}$,
2. arrows,

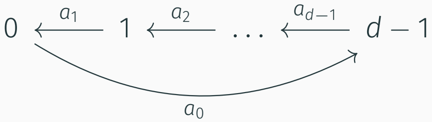
$$Q_1 = \{a_i \mid i \in \mathbb{Z}/d\mathbb{Z}\}$$

   such that $s(a_i) = i$ and $t(a_i) = i - 1$,

3. subject to the following relations

$$a_{i-l+1} \circ \ldots \circ a_{i-1} \circ a_i = 0$$

   for all $i \in \mathbb{Z}/d\mathbb{Z}$.



Nonsemisimple indecomposable $\text{Rep}(H_l)-$module categories

Equivalent Category of Quiver Representation

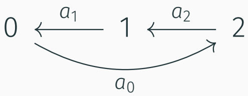
The category $\text{Rep}(\mathcal{Q})$ is equivalent to the module category $\text{Mod}_{\text{Rep}(H_l)}(A)$ where $A = \Bbbk^{G/H}$ and H is the unique subgroup of G of size d.

Properties of $\text{Rep}(\mathcal{Q})$

The category $\text{Rep}(\mathcal{Q})$ is nonsemisimple indecomposable.

Example ($l = 6$ and $d = 3$)

Let $l = 6$ and $d = 3$, that is, $G \cong \mathbb{Z}/6\mathbb{Z}$, $H \cong \mathbb{Z}/3\mathbb{Z}$ and $G/H \cong \mathbb{Z}/2\mathbb{Z}$. The algebra $A = \Bbbk^{G/H} \cong \Bbbk^{\mathbb{Z}/2\mathbb{Z}}$.
Consider the quiver Q:

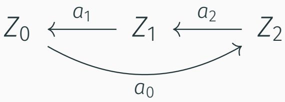


subject to the relations I:

$$(a_{i-2} \circ a_{i-1} \circ a_i)^2 = 0, \qquad \forall i \in \mathbb{Z}/3\mathbb{Z}$$

Example ($l = 6$ and $d = 3$)

Given a representation over (Q, I)



For each $i \in \mathbb{Z}/3\mathbb{Z}$ and $j \in \mathbb{Z}/2\mathbb{Z}$, set $\mathcal{Z}_{(i,j)} = Z_i$. Then the vector space

$$\mathcal{Z} \cong \bigoplus_{i \in \mathbb{Z}/3\mathbb{Z}, j \in \mathbb{Z}/2\mathbb{Z}} \mathcal{Z}_{(i,j)}$$

carries a left H_l right A module structure.

Example ($l = 6$ and $d = 3$)

The category $\mathrm{Rep}(Q, I)$ is nonsemisimple. Simple objects are of the form

$$\begin{tikzcd} \Bbbk & 0 \arrow[l, "0"'] & 0 \arrow[l, "0"'] \arrow[from=1-1, to=1-3, bend right, "0"'] \end{tikzcd}$$

and its cyclic permutations. An example of an indecomposable object in $\mathrm{Rep}(Q, I)$ is

$$\begin{tikzcd} \Bbbk & \Bbbk \arrow[l, "1"'] & 0 \arrow[l, "0"'] \arrow[from=1-1, to=1-3, bend right, "0"'] \end{tikzcd}$$

Suppose $A = \Bbbk^{G/H} = \langle e_i \rangle$ and $B = \Bbbk^{G/K} = \langle f_j \rangle$ with H and K of size d and d' respectively. Let V be an $(A, B)-$bimodule in Rep_{H_l}. Then we have

$$g(e_i V f_j) = e_{i+1} V f_{j+1}$$

and g cycles through $\mathrm{lcm}\left(\frac{l}{d}, \frac{l}{d'}\right)$ many components before returning to the same one.

Let $\mathcal{Q}$ be a quiver as above with $\gcd(d, d')$ many vertices.

Equivalent Quiver Category

The category $\mathrm{Bimod}_{\mathcal{C}}(A, B)$ is equivalent to $\mathrm{Rep}(\mathcal{Q})$.

The category $\mathrm{Mod}_{\mathrm{Rep}(H_l)}(A(d,\lambda))$ is semisimple and the goal is to find the canonical decomposition of objects into simple subobjects.

Let $V \in \mathrm{Mod}_{\mathrm{Rep}(H_l)}(A(d,\lambda))$ and $K = \ker(x)$.

The subspace K is invariant under the action of H_l and hence decomposes as eigensubspaces of $g^t \in H_l$.

$$K = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} K^{(i)}$$

Define

$$\overline{K^{(i)}} := \bigoplus_{r \in \mathbb{Z}/l\mathbb{Z}} K^{(i)} y^r$$

The subspace $\overline{K^{(i)}}$ is invariant under the action of both H_l and $A(d,\lambda)$.

Canonical Decomposition

The module V has the canonical decomposition

$$V = \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \overline{K^{(i)}}$$

into subobjects of $\mathrm{Mod}_{\mathrm{Rep}(H_l)}(A(d,\lambda))$.

Suppose $A = A(d, \lambda)$ and $B = A(d', \lambda')$. The module categories $(A)\mathrm{Mod}_{\mathcal{C}}$ and $\mathrm{Mod}_{\mathcal{C}}(B)$ are semisimple.

Semisimplicity

The category $\mathrm{Bimod}_{\mathcal{C}}(A, B)$ is semisimple.

The category $\mathrm{Bimod}_{\mathcal{C}}(A, B)$ is non-semisimple in the mixed case.

A similar description of categories of $(A, B)-$bimodules over generalized Taft algebras can be given.

Future Directions

Currently exploring:

- Interpret the cyclic quivers and their length-l relations as Galois coverings of the Taft module categories.
- Use these bimodules to study module categories and tensor algebras over $Rep(U_q(\mathfrak{b}))$

Thank You!

Questions?

Thank you for your attention!

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