Module categories over representations of Taft algebras

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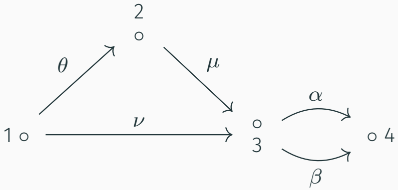
Quiver

A quiver Q is a directed graph, formally described by:

1. a collection of vertices Q_0,
2. a collection of edges (or arrows) Q_1,
3. maps $s : Q_1 \to Q_0$ and $t : Q_1 \to Q_0$ describing the source and target of an arrow.

Quiver

A quiver is a directed graph with a collection of vertices, a collection of arrows, and maps describing the initial and terminal vertices of an arrow.



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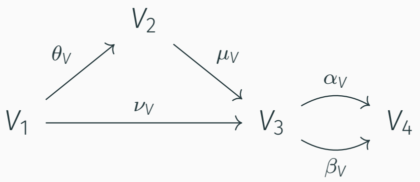
Representations of a Quiver

A representation V of a quiver Q is the following collection of data:

1. for every vertex $x \in Q_0$, a vector space V_x,
2. for every arrow $x \xrightarrow{\alpha} y$ a linear map $\alpha_V : V_x \to V_y$.

Example:



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Morphisms of Representations

A morphism of representations $f : V \to W$ is a collection of linear mas $f_x : V_x \to W_x$ such that the following diagram commutes

$$\begin{array}{ccc} V_x & \xrightarrow{\alpha_V} & V_y \\ {\scriptstyle f_x}\downarrow & & \downarrow{\scriptstyle f_y} \\ W_x & \xrightarrow{\alpha_W} & W_y \end{array}$$

for all arrows $x \xrightarrow{\alpha} y$ in the quiver Q.

Path Algebra

The path algebra $\Bbbk Q$ of a quiver Q is the algebra generated by paths in Q (including paths of length 0, that is, orthogonal idempotents e_x for each vertex $x \in Q_0$).

The algebra $\Bbbk Q$ is an associative algebra with unit $1 = \sum e_x$.

The vector space $\Bbbk Q_1$ generated by arrows in Q is a bimodule over the algebra $\Bbbk Q_0$ generated by the orthogonal idempotents.

The path algebra $\Bbbk Q$ can be viewed as the tensor algebra

$$\Bbbk Q_0 \oplus \Bbbk Q_1 \oplus \Bbbk Q_2 \oplus \dots$$

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Theorem [Sch16]

The category of representations of a quiver Q is equivalent to the category $\Bbbk Q - \text{mod}$ of left modules over the path algebra $\Bbbk Q$.

Theorem [ASS06]

Let A be a finite dimensional associative algebra.

- A is Morita equivalent to some basic algebra A_b
- (Garbriel's theorem) There exists a quiver Q and an admissible ideal $\mathcal{I} \subset \Bbbk Q$ such that $A_b \cong \Bbbk Q/\mathcal{I}$.

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Restatement of the above result:

Theorem

Let A be an algebra in $\text{vec}_{\Bbbk}$, the category of finite dimensional $\Bbbk-$vector spaces. There exists a quiver Q and an admissible ideal $\mathcal{I} \subset \Bbbk Q$ such that the categories $A -$ mod and $\Bbbk Q/\mathcal{I} -$ mod are equivalent.

Are there any interesting generalizations of this result to arbitrary tensor categories or module algebras over finite dimensional Hopf algebras?

Module algebra (left H–action)

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Let H be a Hopf algebra over $\mathbb{k}$ with coproduct $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ and counit ε. A unital $\mathbb{k}$–algebra A with a left H–action $h \triangleright a$ is a left H–module algebra if

$$h \triangleright (ab) = \sum (h_{(1)} \triangleright a)\,(h_{(2)} \triangleright b), \qquad h \triangleright 1_A = \varepsilon(h)\,1_A$$

$$(\forall\, h \in H,\ a, b \in A).$$

Equivalently: the multiplication $m : A \otimes A \to A$ and unit $\eta : \mathbb{k} \to A$ are H–linear for the H–action on $A \otimes A$.

Smash product of an H–module algebra

Let H be a Hopf algebra over $\Bbbk$ with coproduct $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ and counit ε. Let A be a left H–module algebra, i.e.

$$h \triangleright (ab) = \sum (h_{(1)} \triangleright a)\,(h_{(2)} \triangleright b) \quad \text{and} \quad h \triangleright 1_A = \varepsilon(h)\,1_A$$

$$(\forall\, a, b \in A,\ \ h \in H).$$

Definition (smash product $A\#H$)

As a vector space, $A\#H := A \otimes H$ (write $a\#h$ for $a \otimes h$), with unit $1_A\#1_H$ and multiplication

$$(a\#h)(b\#\Bbbk) \;=\; \sum a\,(h_{(1)}\triangleright b)\;\#\;h_{(2)}\Bbbk, \qquad a, b \in A,\ \ h, \Bbbk \in H.$$

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Canonical embeddings and cross relation:

$$\iota_A : A \to A\#H,\ a \mapsto a\#1, \qquad \iota_H : H \to A\#H,\ h \mapsto 1\#h,$$

and

$$(1\#h)(a\#1) = \sum (h_{(1)} \triangleright a)\#h_{(2)}.$$

Special case ($H = kG$). If H is a group algebra and G acts on A, then

$$(a\#g)(b\#h) = a\,(g \triangleright b)\#gh.$$

Theorem (Equivalence via smash product) [Eti+15]

There is an equivalence of categories

$$\mathrm{Mod}_{\mathrm{Rep}(H)}(A) \simeq \mathrm{Mod}(A\#H),$$

natural in (A, H).

Functors.

- $F : \mathrm{Mod}_{\mathrm{Rep}(H)}(A) \to \mathrm{Mod}(A\#H)$: for an H-equivariant A-module M,
$$(a\#h) \cdot m := a \cdot (h \triangleright m).$$
- $G : \mathrm{Mod}(A\#H) \to \mathrm{Mod}_{\mathrm{Rep}(H)}(A)$: for a left $A\#H$-module N,
$$a \cdot n := (a\#1)\, n, \qquad h \triangleright n := (1\#h)\, n.$$

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Definition

The Taft Hopf algebra $T_\ell(\zeta)$ is the algebra over $\Bbbk$ generated by g, x with relations

$$g^\ell = 1, \qquad x^\ell = 0, \qquad gx = \zeta xg,$$

with Hopf structure

$$\Delta(g) = g \otimes g, \quad \Delta(x) = x \otimes 1 + g \otimes x, \qquad \varepsilon(g) = 1,\ \varepsilon(x) = 0,$$

$$S(g) = g^{-1}, \qquad S(x) = -g^{-1}x.$$

Remarks. $\dim_\Bbbk T_\ell(\zeta) = \ell^2$ (basis $\{g^i x^j\}_{0 \le i,j < \ell}$); it is neither commutative nor cocommutative for $\ell > 1$.

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Theorem (Taft–quiver presentation)

Let $A \simeq \Bbbk Q/I$ be a finite-dimensional basic $\Bbbk$–algebra with a left $T_\ell(\zeta)$–action. Then

$$A\#T_\ell \;\text{ is Morita equivalent to }\; \Bbbk Q_T/I_T,$$

where (Q_T, I_T) is determined as follows.

- **Vertices:** $(Q_T)_0 = \{(i,r) \mid i \in Q_0,\ r \in \mathbb{Z}/\ell\mathbb{Z}\}$, where r indexes the g–layer.
- **Horizontal arrows:** for each arrow $a : i \to j$ of Q and each r, add $a^{(r)} : (i,r) \to (j,r)$.
- **Vertical arrows (x):** for each i and r, add $\delta_i^{(r)} : (i,r) \to (g \cdot i,\ r+1)$.

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Relations I_T (transport + Taft cross-relations):

(A) Layerwise transport: copy every relation of I into each layer r by replacing paths with their r–labelled copies.

(B) Vertex x–action: for all i, r,

$\delta_i^{(r)} = (x{\cdot}e_i)^{(r)}$ (a linear comb. of paths $(i, r) \to (g \cdot i, r{+}1)$).

(C) Skew–Leibniz on arrows: for each $a : i \to j$ and r,

$$\delta_j^{(r)}\, a^{(r)} - (g \cdot a)^{(r+1)}\, \delta_i^{(r)} = (x \cdot a)^{(r)}.$$

(D) Taft powers: indices are taken mod ℓ (from $g^\ell = 1$), and any vertical path of length ℓ vanishes (from $x^\ell = 0$); the relation $xg = \zeta\, gx$ is reflected in (C).

Moreover, with $e = \sum_{i \in Q_0} (i, 0)$ there is an isomorphism $\Bbbk Q_T/I_T \cong e(A\#T_\ell)e$, hence $A\#T_\ell \simeq_M \Bbbk Q_T/I_T$.

Let A, B be finite-dimensional T_ℓ–module algebras. Choose full idempotents

$$e = \sum_{i \in Q_0^A} e_i \subset A, \qquad f = \sum_{j \in Q_0^B} f_j \subset B$$

so that

$$eAe \cong \Bbbk Q_A / I_A, \qquad fBf \cong \Bbbk Q_B / I_B$$

are basic and admit Taft–quiver models

$$e(A \# T_\ell)e \cong \Bbbk Q_T^A / I_T^A, \quad f(B \# T_\ell)f \cong \Bbbk Q_T^B / I_T^B.$$

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Theorem (Description of equivariant bimodules)

The category of T_ℓ–equivariant (A, B)-bimodules is equivalent to representations of the Taft biquiver whose vertices are pairs of Taft–quiver vertices

$$(i, r \mid j, s) \in (Q_T^A)_0 \times (Q_T^B)_0$$

and whose arrows come from left arrows in Q_T^A and right arrows in Q_T^B. In practice, an equivariant (A, B)-bimodule is just a matrix of spaces

$$M_{i,r;\,j,s} = e_{i,r} M f_{j,s}$$

with linear maps along these arrows satisfying the Taft–quiver relations.

What's next?

Currently exploring:

- Generalizations to other finite dimensional Hopf algebras.
- Extend these results to module categories and algebra objects in $Rep(U_q(\mathfrak{b}))$

Thank You!

Questions?

Thank you for your attention!

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