Research Statement

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1 Introduction: From quivers to tensor categories

My research lies at the intersection of representation theory, noncommutative algebra, and tensor categories. A central theme is to bridge the gap between abstract categorical frameworks and concrete combinatorial tools. Classical representation theory of finite-dimensional algebras is built on quivers, path algebras, and Morita equivalence. I aim to generalize this quiver-theoretic approach to the setting of module and bimodule categories over non-semisimple tensor categories.

Gabriel's theorem shows that a finite-dimensional algebra of finite representation type is Morita equivalent to the path algebra of a bound quiver. This result is pivotal because it translates abstract module-theoretic problems into concrete linear-algebraic data. Over the last several decades, attention has shifted from algebras over vector spaces to algebras internal to monoidal categories, and from ordinary module categories to module categories over finite tensor categories.

Finite tensor categories and their module categories were systematically developed by Etingof and Ostrik [EO04]. Andruskiewitsch and Mombelli [AM07] provided a classification of module categories over finite-dimensional Hopf algebras via comodule algebras. Montgomery and Schneider [MS01] initiated a detailed study of Hopf actions on finite-dimensional algebras, particularly involving Taft algebras and their doubles. Kinser and Walton [KW16], and later Kinser and Oswald [KO21], gave quiver-theoretic classifications of actions of pointed Hopf algebras and certain small quantum groups on path algebras. In a parallel direction, Etingof, Kinser, and Walton [EKW21] developed the theory of tensor algebras in finite tensor categories, providing a conceptual way to encode algebra objects and their module categories inside a tensor category.

While the classification of module categories over semisimple tensor categories (fusion categories) is relatively well understood, the non-semisimple landscape remains largely unexplored. At the same time, work of Elias and Heng [EH24] on fusion quivers shows how effective a good combinatorial model can be in the semisimple case. My research develops an analogous combinatorial language in the non-semisimple setting. More precisely, I seek a Gabriel-style combinatorial calculus for finite tensor categories and their module and bimodule categories, with an emphasis on explicit and computable models.

Concretely, I work with Taft algebras and related Hopf algebras as canonical non-semisimple test cases. My results provide (i) a combinatorial description of module categories over Taft algebras in terms of graded and "thickened" quivers, and (ii) a combinatorial description of bimodule categories, and therefore of Morita equivalences, between such module categories. Both results are compatible with the classical frameworks of EO and AM, and they interact naturally with the tensor-algebra viewpoint of Etingof–Kinser–Walton.

2 Current research: combinatorics of Taft algebras

2.1 Taft algebras as a non-semisimple example

Taft algebras T_ℓ are pointed Hopf algebras generated by a grouplike element and a skew-primitive element, with relations $g^\ell = 1$, $x^\ell = 0$, and $xg = \zeta gx$ for a primitive ℓ-th root of unity ζ. They are among the simplest non-semisimple Hopf algebras. Their representation categories $\mathrm{Rep}(T_\ell)$ are finite tensor categories in the sense of [EO04].

The classification of module categories over these and related tensor categories is already available at an abstract level. Etingof–Ostrik [EO04] classify exact module categories over finite tensor categories via algebra objects, and Andruskiewitsch–Mombelli [AM07] classify module categories over certain finite-dimensional Hopf algebras via comodule algebras. These are fundamental structural results that describe all module categories in terms of algebraic objects inside the tensor category or the Hopf algebra.

My contribution complements this structural picture with a **quiver-theoretic and diagrammatic model**. I provide explicit combinatorial data that realizes the EO and AM classifications in the spirit of Etingof–Kinser–Walton's tensor-algebra framework and the quiver-level descriptions of Hopf actions in [KW16; KO21]. This converts existence and classification theorems into calculable structures and reveals covering-theoretic and periodic patterns that are difficult to access from the algebra-object viewpoint alone.

2.2 Theorem A: from EO/AM classification to explicit quivers

The first main result provides an explicit Gabriel-type theorem for module categories over $\mathrm{Rep}(H_\ell)$, where H_ℓ denotes the Taft Hopf algebra. An indecomposable exact module category $\mathcal{M}$ over $\mathrm{Rep}(H_\ell)$ can be encoded by a quiver endowed with a compatible $\mathbb{Z}/\ell\mathbb{Z}$-grading and a "thickening" structure that records the arrangement of projective covers and Loewy layers along a discrete cylinder associated with the Taft grading. Vertices correspond to simples in $\mathcal{M}$, and edges reflect generators of internal Homs and extensions compatible with the H_ℓ-action.

Theorem 2.1 (Module Categories)**.** *Let H_ℓ be the Taft Hopf algebra over an algebraically closed field of characteristic* 0*, and let $\mathcal{C} = \mathrm{Rep}(H_\ell)$. For every divisor $d \mid \ell$ let $H \le \langle g \rangle$ be the subgroup of order d. Then:*

1. ***Non-semisimple block.*** *The module category $\mathrm{Mod}_{\mathcal{C}}(k[\langle g \rangle / H])$ is indecomposable, exact, and non-semisimple with d simple objects. It is equivalent to the category of representations of the cyclic quiver with vertex set $\mathbb{Z}/d\mathbb{Z}$, arrows*

$$a_i : i \to i - 1,$$

and length-ℓ nilpotency relations

$$a_{i-\ell+1} \cdots a_i = 0 \quad \textit{for all } i.$$

2. ***Semisimple family.*** *For each $\lambda \in k$, let $A(d,\lambda)$ be the H_ℓ-module algebra generated by $k[\langle g\rangle/H]$ and an element y satisfying*

$$x(y) = 1, \quad g(y) = q^{-1}y, \quad y^\ell = \lambda \cdot 1.$$

Then $\mathrm{Mod}_{\mathcal{C}}(A(d,\lambda))$ is semisimple with d simple objects. Every object V in this category decomposes canonically as

$$V \cong \bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}} K(i), \qquad K(i) = \bigoplus_{r\in\mathbb{Z}/\ell\mathbb{Z}} K(i)_r y^r,$$

where $K = \ker(x)$ and $K(i)$ is the eigenspace of the g-action with eigenvalue q^i.

Connection to EO and AM. This theorem makes the EO classification of exact module categories explicit in the Taft setting by providing an actual cyclic quiver model and explicit Loewy-length data. It also refines the AM comodule-algebra classification by turning the structural description into a fully computable quiver calculus. The result reveals a cylindrical geometry: simples and projectives sit on a discrete cylinder whose longitudinal and angular coordinates record Loewy length and group grading.

2.3 Theorem B: explicit quiver classification of Morita equivalences

Module categories over finite tensor categories form a 2-category whose 1-morphisms are bimodule categories. For Taft algebras, many natural tensor functors arise from Hopf actions on path algebras and from smash or crossed products, as in the work of Montgomery–Schneider [MS01], Kinser–Walton [KW16], and Kinser–Oswald [KO21].

Extending the combinatorial description to bimodule categories, the following result shows that these correspond to pairs of graded, thickened quivers with additional connecting data. This gives an explicit, quiver-level criterion for Morita equivalence.

Theorem 2.2 (Bimodule Categories)**.** *Let $\mathcal{C} = \mathrm{Rep}(H_\ell)$ and consider the indecomposable $\mathcal{C}$-module algebras*

$$A = k[\langle g\rangle/H], \qquad B = k[\langle g\rangle/K],$$

where $|H| = d$ and $|K| = d'$. Then:

1. ***Non-semisimple block case.*** *The bimodule category $\mathrm{Bimod}_{\mathcal{C}}(A,B)$ is equivalent to the representations of the cyclic quiver with*

$$\gcd(d,d') \quad \textit{vertices.}$$

2. ***Semisimple case.*** *If $A = A(d,\lambda)$ and $B = A(d',\lambda')$, then $\mathrm{Bimod}_{\mathcal{C}}(A,B)$ is semisimple.*

3. ***Mixed case.*** *If $A = k[\langle g\rangle/H]$ and $B = A(d',\lambda')$ (or the reverse), then $\mathrm{Bimod}_{\mathcal{C}}(A,B)$ is non-semisimple. It is described by the same cyclic quiver with $\gcd(d,d')$ vertices.*

This provides a precise, hands-on description of bimodule categories, giving **explicit quiver models** for all Morita contexts between Taft-module categories. It clarifies when bimodule categories are semisimple and when they carry non-semisimple extension structure, and it is compatible with Hopf actions on path algebras studied in [MS01; KW16; KO21].

2.4 Enriched Gabriel diagrams and internal Homs

A natural framework for these constructions is an enriched Gabriel diagram for a finite tensor category $\mathcal{C}$ or for a module category $\mathcal{M}$. Vertices represent simples, edges represent internal Homs or bimodule objects, and gradings encode tensoring with distinguished objects and the placement of projectives. For $\mathcal{C} = \mathrm{Rep}(T_\ell)$, these diagrammatic structures recover both Theorem A and Theorem B. They are also compatible with the tensor-algebra viewpoint of [EKW21], which suggests that enriched quivers can serve as a graphical model for tensor algebras and module categories in finite tensor categories.

3 Future research program

3.1 From Taft algebras to general non-semisimple tensor categories

The Taft case reveals structural patterns such as graded coverings, cylindrical periodicity, and thickening of quivers. I plan to extend these ideas to pointed and weakly group-theoretical tensor categories, where group gradings and crossed-product structures remain central. I also plan to study how enriched Gabriel diagrams behave under exact sequences of tensor categories and how representation type and homological properties manifest in the combinatorial data.

3.2 Labelled quivers and the fusion-quiver limit

In the semisimple setting, fusion categories admit combinatorial models via fusion graphs or quivers. Elias and Heng [EH24] have recently classified fusion quivers of finite type. I aim to develop a labelled-quiver formalism that specializes to fusion quivers in the semisimple case and extends naturally to non-semisimple tensor categories. This should lead to a unified diagrammatic calculus with applications in representation theory and topological quantum field theory.

3.3 Autoequivalence-twisted Cartesian products of quivers

Cartesian products of quivers model tensor products of path algebras. In the presence of gradings, Hopf actions, and autoequivalences, one encounters twisted Cartesian products corresponding to smash and crossed products and to tensor algebras in finite tensor categories. Throughout this work, the term "twist" refers to a modification induced by a fixed automorphism or autoequivalence, not deformation in the homological sense.

I plan to study autoequivalence-twisted products $Q\Box_\phi Q'$, the corresponding skew tensor products of path algebras, and their representation theory. A further goal is to express these structures purely in the enriched-quiver language, linking them directly to tensor algebras in finite tensor categories.

3.4 Computational algebra and visualization

I plan to develop computational tools in Python and Sage to generate and manipulate quivers, coverings, thickenings, and autoequivalence-twisted products, together with their representation categories. A key aim is to support the **visualization** of module categories, bimodule categories, Green rings, and adjacency matrices. Visual representations often reveal patterns, periodicities, and symmetries that are difficult to detect from algebraic formulas alone. They also provide an accessible entry point for collaborators and students with different backgrounds.

These tools will be used to test conjectures, explore families of examples, and produce diagrams that make the geometry of quantum symmetries more tangible. They will also support student-led projects at the interface of algebra, tensor categories, and computation.

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